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GUICHARDET–WIGNER PSEUDOCHARACTER ON THE UNIVERSAL COVERING GROUP OF SL(2, ℝ) HAS A NONTRIVIAL RESTRICTION TO THE PREIMAGE OF SL(2, ℤ)

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Abstract. We prove that the restriction of a Guichardet–Wigner pseudocharacter on the universal covering group $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$ to the complete preimage H of $\mathrm{SL}(2,\mathbb{Z})$ in G is a nontrivial pseudocharacter of H.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters and quasicharacters, see [1]–[5].

Let G be a group and let $\mathrm{BD}(G)$ be the vector space of real-valued functions f on G for which the function $\delta f\colon G \times G$ defined by the rule

$$(\delta f)(hk) = f(hk) - f(h) - f(k), \qquad h, k \in G, \tag{1}$$

is bounded. The elements of $\mathrm{BD}(G)$ are briefly called quasicharacters on G. A quasicharacter f on G is called a pseudocharacter on G if

$$f(g^n) = nf(g) \quad \text{for all} \quad g \in G \quad \text{and} \quad n \in \mathbb{N}.$$

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To every quasicharacter F on a group G there corresponds a pseudocharacter f defined by the rule

$$f(g) = \lim_{n\to\infty} n^{-1} F(g^n), \qquad g \in G,$$

where the limit exists always [1].

For the Guichardet–Wigner pseudocharacters on connected simply connected Hermitian symmetric simple Lie groups, see [3, 5].

§ 2. Preliminaries

Now let G be the universal covering group $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$. As is well known [3, 5], there is a unique (up to a nonzero numerical factor) pseudocharacter on the group G, namely, the so-called Guichardet–Wigner pseudocharacter f, which is the pseudocharacter corresponding (see [1]–[4]) to the quasicharacter F on G defined in one of the natural parametrizations of G (see [6]) by assigning φ to an element g of the form

$$g(\varphi, a, n) = (\varphi, u_\varphi a n),$$

where $\varphi \in \mathbb{R}$, u_φ is the plane rotation by the angle φ, and $u_\varphi a n$ is the Iwasawa decomposition of the element $g \in G$. Thus, in the above notation,

$$F(g(\varphi, a, n)) = \varphi.$$

The quotient group $L = \mathrm{SL}(2,\mathbb{R})$ of G by the central subgroup of index 2 has no nontrivial pseudocharacters [1]–[4] and has a subgroup $R = \mathrm{SL}(2,\mathbb{Z})$ whose quotient by the center is equivalent to the free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and hence has infinitely many linearly independent pseudocharacters (including the Rademacher pseudocharacter, see [1]).

For this reason, it is of interest to find out whether or not the complete preimage H of R in G has nontrivial pseudocharacters. In this note, we answer this question affirmatively.

§ 3. Main result

Theorem. *The restriction of the Guichardet–Wigner pseudocharacter on G to H is a nontrivial* (*nonzero*) *pseudocharacter on H.*

Proof. Since $f - F$ is bounded [1]–[4], and a bounded pseudocharacter is zero [1]–[4], it follows that, to prove the theorem, it suffices to prove that the restriction of F to H is unbounded.

Suppose not. Let the restriction of F to H be bounded.

Obviously, the homogeneous space G/H is naturally isomorphic to the homogeneous space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. The latter is naturally isomorphic to the space of linear elements on the fundamental domain in the upper half-plane (the exterior of the unit disk in the strip $|\Re z| \in (-1/2, 1/2])$ [7].

For every element $g \in G$, we can find an $h \in H$ such that h returns the image of the horizontal positively directed linear element at the point $z = i$ to the fundamental domain.

Then we find an n (moving the point on the half-plane horizontally) taking the point to the positive part of the ordinate axis.

Next we use an a (multiplying any point by a positive number) to move the intermediate result to the point $z = i$.

Finally, we use a rotation u by at most π to return the linear element at $z = i$ to the positive direction of the horizontal.

We thus obtain

$$ghanu = e_G$$

and

$$g = u^{-1}n^{-1}a^{-1}h^{-1},$$

where F is bounded on the subgroups $\{a\}$, $\{n\}$, and H and on the family $\{u_\varphi, |\varphi| \leq \pi$, which contain the factors on the right-hand side of the last equation, and hence F is bounded on G, which is impossible, since the pseudocharacter f is nontrivial. This contradiction shows that the restriction of F to H is unbounded, and the corresponding pseudocharacter (the restriction of the Guichardet–Wigner pseudocharacter on G to H) is nonzero.

§ 4. Comments

The above theorem poses a natural problem to evaluate the restriction in question.

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References

1. A. I. Shtern, *Quasisymmetry. I*, Russ. J. Math. Phys. **2** (1994), no. 3, 353–382.
2. A. I. Shtern, *Kazhdan–Milman Problem for Semisimple Compact Lie Groups*, Russian Math. Surveys **62** (2007), no. 1, 113–174.
3. A. I. Shtern, *A Version of van der Waerden's Theorem and a Proof of Mishchenko's Conjecture on Homomorphisms of Locally Compact Groups*, Izv. Math. **72** (2008), no. 1, 169–205.
4. A. I. Shtern, *Locally Bounded Finally Precontinuous Finite-Dimensional Quasirepresentations of Locally Compact Groups*, Sb. Math. **208** (2017), no. 10, 1557–1576.
5. A. I. Shtern, *Finite-Dimensional Quasi-Representations of Connected Lie Groups and Mishchenko's Conjecture*, J. Math. Sci. **159** (2009), 653–751.
6. P. J. Sally, Jr., *Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group of* $\mathrm{SL}(2,\mathbb{R})$, Mem. Amer. Math. Soc. No. 69, American Mathematical Society, Providence, RI, 1967.
7. I. M. Gel'fand, M. I. Graev, I. I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Functions*, Academic Press, Inc., Boston, MA, 1990.

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